Wigner-Dyson Statistics from the Keldysh σ-Model

Alexander Altland[1], and Alex Kamenev[2]
[1] *Theoretische Physik III, Ruhr-Universität Bochum, 44780, Germany.*
[2] *Department of Physics, Technion, Haifa 32000, Israel.*
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The level statistics of disordered metallic grains with broken time reversal invariance is obtained from a saddle point analysis of the Keldysh nonlinear σ–model.

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In the past two decades, the field theory of disordered electronic systems, a matrix version of the nonlinear σ–model, has attracted a great deal of theoretical attention [1–6]. The σ-model provides the perhaps most rigorous and comprehensive theoretical framework by which interacting and non–interacting disordered electronic systems can be explored. Three different, albeit closely related versions of this model exist: The supersymmetric (SUSY) [1], replica [2] and dynamic or Keldysh [3] σ-model. Due to different microscopic starting points these three formulations cover partly complementary application areas: whereas SUSY is tailor made to the analysis of non-perturbative phenonmena, the replica, and in particular the Keldysh formalism are strong at problems involving interactions – about which SUSY has nothing to say. This constellation amounts to some theoretical vacuum because sooner or later non-perturbative interaction related problems will come into focus.

An important step towards improving the situation was made recently [4] (first within the framework of SUSY) when it became clear that the large energy asymptotic of Wigner-Dyson (WD) level statistics – *the* non-perturbative prototype problem – may be obtained from a careful analysis of the saddle point structure of the σ–model (in contrast to a full integration over the field manifold which is out of question for the non-SUSY representants). Subsequently these ideas were adopted to the replica σ–model [5], which proved to be useful for the solution of simple interacting problems [6]. Yet, the application spectrum of the replica theory, excluding non-equilibrium phenomena, is significantly narrower than that of its Keldysh counterpart. Furthermore, replica analyses categorically rely on a formidable analytic continuation procedure, which tends to obscure the underlying phyiscs and essentially complicates practical applications. It is therefore desirable to formulate a prescription whereby non-perturbative information can be obtained from the conceptually more transparent Keldysh formalism. This is the subject of the present Letter.

All the theories mentioned above share the normalization condition $Z = 1$, where Z is the functional partition function. It is this universal normalzation property which makes the models useful in the analysis of disorder properties [1]. The normalization in turn is due to a global internal symmetry underlying the model: supersymmetry, replica permutation symmetry or, within the Keldysh formulation, a symmetry principle related to the causality of the theory. Building on these structures, the saddle point analysis of any σ–model must consist of three parts: (i) identification of *all* saddle points and their action, (ii) proof that $Z = 1$ holds within the stationary phase scheme (iii) saddle point evaluation of any specific correlation function. Using WD spectral statistics as a example we are going to apply this program to the Keldysh model of unitary (broken time reversal invariance) symmetry. The generalization to different symmetries and observables will be discussed elsewhere [7].

We wish to compute the two-point correlation function $R(\epsilon_1,\epsilon_2) = \Delta^2\langle\langle\rho(\epsilon_1)\rho(\epsilon_2)\rangle\rangle = \Delta^2/(2\pi^2)\,\Re\,\langle\langle\,\mathrm{tr}\,\{G(\epsilon_1^+)\}\,\mathrm{tr}\,\{G(\epsilon_2^-)\}\rangle\rangle$ of the density of states $\rho(\epsilon)$ for a model of free electrons with broken time reversal invariance and subject to a random potential. Here Δ is the mean level spacing and $\langle\langle\ldots\rangle\rangle$ stands for a (cumulative) disorder average. Our starting point is the low energy effective partition function [3] $Z = \int \mathcal{D}Q e^{iS[Q]}$, where

$$iS[Q] = -\frac{\pi}{4\Delta}\int\frac{d^d\mathbf{r}}{L^d}\,\mathrm{tr}\,\{D\partial Q\partial Q + 4i\hat{\epsilon}Q\}\,, \qquad (1)$$

D is the diffusion constant and L the system size. The matrix–valued field $Q = \{Q^{l,l'}_{\epsilon,\epsilon'}\}$ acts in a product space defined through an index $l = 1,2$ labelling the forward and backward Keldysh time contour, $\mathcal{C}^l = \{t \in (\mp\infty,\pm\infty)\}$, and the energy indices, ϵ, Fourier conjugate to the time variables t on $\mathcal{C}^l$. For all what follows it will be convenient to discretize the (a priori continuous) energy variables in units of some small spacing δ_ϵ. Since the application range of the effective action is limited to energies $|\epsilon| < \tau^{-1}$ [2,3], this manipulation implies that the Q's become matrices of finite dimension $2K$, where $K = 2/(\delta_\epsilon\tau)$. The internal structure of these matrices is defined through (a) the constraint $Q^2(\mathbf{r}) = \mathbb{1}$ and (b) Hermiticity, $Q^\dagger = Q$. Finally, $\hat{\epsilon} = \mathrm{diag}\,(\hat{\epsilon}^1,\hat{\epsilon}^2)$ is a $2K$-dimensional diagonal matrix where $\hat{\epsilon}^l = \mathrm{diag}(\epsilon_1,\ldots\epsilon_n,\ldots\epsilon_K) - i(-1)^l 0$.

The definition of the low energy effective action (1) implies one more important structural element which is not explicit in the notation: by definition of the trace 'tr', the energy integration/summation over any continuous and analytically benign function of the energy indices ϵ produces zero (e.g. $\mathrm{tr}\,\{\hat{\epsilon}\} = 0$). For lack of better terminology we will refer to this feature as 'causality'. That

the causality criterion is not explicit in the definition of the trace is not just a matter of notational convenience. The crux is that to rigorously retrieve the full causality properties of the microscopic Keldysh partition function, *all* energies, ϵ, including energies in excess of the width of the spectrum of the microscopic Hamiltonian, have to be taken into account. The philosophy behind declaring the causality principle to an intrinsic feature of the effective action (1) is that S merely represents the low energy sector of some larger parent theory, $S + S_{\rm high}$. After the inclusion of the high energy sector $S_{\rm high}$, all energy summations could, in principle, be extended to infinity and the correct spectral structures of the Keldysh partition function would be retrieved.

This anticipation has been at the root of previous effective action formulations of the Keldysh approach, and, needless to say, is of relevance for all matters related to spectral statistics. We will therefore subdivide our analysis into two parts. Taking a pragmatic point of view we will first show how the spectral correlation function can be obtained from the low energy model, Eq. (1), once the causality assumption has been made. In a second part we will then show how causality can be made *manifest*, on the expense of including large energies. We emphasize that part II of the analysis has been included for reasons of conceptual completeness; in practical applications, the large energy sector will normally not play a role.

Part I: The simplest way of computing $R(\epsilon_1, \epsilon_2)$ from the functional integral over the effective action $iS[Q]$, is to couple the energy vector $\hat{\epsilon}$ to sources. This can be done by generalizing $\hat{\epsilon}^l \to \hat{\epsilon}^{l,\kappa} \equiv \hat{\epsilon}^l + \hat{\kappa}^l$, where the energy diagonal matrices $\hat{\kappa}^l = \{\delta_{nn'}\delta_{nn_l}\kappa_l\}$ and ϵ_{n_l} are the discrete energies closest to the arguments ϵ_l, $l = 1, 2$. It is then a straightforward matter to verify that the definition of the Keldysh partition function implies $R(\epsilon_1, \epsilon_2) = \Delta^2/(2\pi^2)\,\mathrm{Re}\,\partial^2_{\kappa_1\kappa_2}\big|_{\kappa=0} Z[\hat{\kappa}] - \mathrm{dis}$, where 'dis' stands for the disconnected part of the functional average. To keep the presentation simple we will focus in the quantitative analysis of this expression on the contribution of the spatial zero mode $Q(\mathbf{r}) \equiv Q = \mathrm{const.}$ (which ought to reproduce WD statistics). The inclusion of the spatially fluctuating modes, which is straightforward and does not introduce conceptually new elements, will be briefly discussed in the end.

The key to understanding the structure of the zero mode integration lies in the observation that the (source-free) effective action $iS[Q] = -i\pi\Delta^{-1}\,\mathrm{tr}\,\{\hat{\epsilon}Q\}$ possesses a multitude of 2^{2K} isolated saddle points. Indeed, any of the (energy diagonal [9]) configurations $\Lambda = \mathrm{diag}\,(\pm 1, \pm 1, \ldots, \pm 1)$ solves the stationary phase equation of the model, $\delta_Q S[Q] = \pi\Delta^{-1}[\hat{\epsilon}, Q] = 0$. In what follows we are going to show that a Gaussian integration around these saddle points produces WD statistics.

To prepare the integration, consider the contribution of any saddle point Λ with $K+p$ entries $+1$ and $K-p$ entries -1. We first re-order these elements (through some global unitary transformation) such that Λ assumes the form $\Lambda = \mathrm{diag}\,(1, \ldots, 1, -1, \ldots -1)$. Next, a set of field configurations weakly fluctuating around Λ is introduced through $Q = T\Lambda T^{-1}$, where the unitary rotation matrices $T = \exp\left(\begin{smallmatrix} 0 & B \\ -B^\dagger & 0 \end{smallmatrix}\right)$, B is a $(K+p)\times(K-p)$-dimensional complex generator matrix, and the block decomposition corresponds to the signature of Λ.

In principle we should now proceed by expanding the action to second order in B and integrate. Fortunately, however, there is no need to carry out this program for all 2^{2K} saddle points explicitly. The reason is that among the entity of saddle points Λ, there is one element $\Lambda_0^{l,l'} \equiv (-1)^{l+1}\delta^{l,l'}$ that plays a distinguished role. What makes Λ_0 special is that, unlike the other Λ's, its structure is compatible with the signature of the imaginary increments of $\hat{\epsilon}$. Building on this feature, previous analyses of the Keldysh σ-model indeed focused on a perturbative expansion around Λ_0 and did not take the other saddle points into account.

As a warm-up, let us outline how an integration around the standard saddle point produces the unit–normalization of the source–free partition function $Z[\hat{\kappa} = 0] = 1$. Substituting $Q = T\Lambda_0 T^{-1}$ into the zero mode action and expanding to second order in B we obtain the quadratic action

$$iS^{(2)}_{\Lambda_0}[B] = \frac{-i\pi}{\Delta}\Big[\mathrm{tr}\{\hat{\epsilon}\Lambda_0\} - \sum_{n,n'}(\epsilon^+_n - \epsilon^-_{n'})|B_{nn'}|^2\Big]. \quad (2)$$

Integration over B then leads to

$$Z_0 = \mathrm{const.} \times e^{-i\pi\Delta^{-1}\mathrm{tr}\{\hat{\epsilon}\Lambda_0\}}F_0\,;\; F_0 = \prod_{n,n'}(\epsilon^+_n - \epsilon^-_{n'})^{-1}$$

as the contribution of Λ_0 to $Z[0]$. Due to the causality property, $\mathrm{tr}\{\hat{\epsilon}\Lambda_0\} = 0$. Similarly, $F_0 = \exp\{-\sum_{nn'}\ln(\epsilon^+_n - \epsilon^-_{n'})\} = \exp\{0\} = 1$, where the presence of the imaginary increments guarantees that the branch cut singularity of the logarithm is not touched. (We re-emphasize that, at this stage, the causality rule has the mere status of a working assumption. In part II we will make up for its precise formulation, and show that the proper ultraviolet extension of the fluctuation determinant fixes the unspecified 'const.' to unity.) Combining factors, we find $Z_0 = 1$. As a corollary we remark that $Z_0 = 1$ implies vanishing of the total contribution of all *non*–standard saddle points to $Z[0]$. Before turning to these other saddle points, let us discuss the contribution R_0 of the standard saddle point to the spectral correlation function. A straightforward expansion of $Z[\hat{\kappa}]$ to first order in κ_1 and κ_2 yields $R_0 = \frac{1}{2}\langle\sum_{nn'}B_{n_1 n}B^\dagger_{nn_1}B^\dagger_{n_2 n'}B_{n'n_2}\rangle_B$, where $\langle\ldots\rangle_B$ stands for the Gaussian average over the action (2). Integration over B then leads to $R_0 = 1/(2s^2)$, where $s = \pi\omega^+/\Delta$ and $\omega^+ \equiv \epsilon^+_{n_1} - \epsilon^-_{n_2}$ [11].

We next turn to the discussion of the other saddle points. In fact we will focus on just a single non-standard saddle point $\tilde{\Lambda}$, namely the one where the signs of the two entries corresponding to our reference energies $\epsilon_{n_{1,2}}$, are flipped: $\tilde{\Lambda} \equiv \Lambda_0 - 2\delta^{l1}\delta_{nn_1} + 2\delta^{l2}\delta_{nn_2}$. From the analysis of $\tilde{\Lambda}$, the role of all the other saddle points will become

clear. The re-ordering needed to bring $\tilde{\Lambda}$ into the canonical form implies that the action $iS^{(2)}_{\tilde{\Lambda}}$ differs from $iS^{(2)}_{\Lambda_0}$ in two respects: first, the contribution of the saddle point itself $iS[\tilde{\Lambda}] = -i\pi\Delta^{-1}\,\mathrm{tr}\,\{\hat{\epsilon}\tilde{\Lambda}\} = i\pi\Delta^{-1}2\omega^{+}$ no longer vanishes. Second, in the fluctuation contribution, the two energy arguments ϵ_{n_2} and ϵ_{n_1} are exchanged. Given these structural changes we find it more convenient to reverse the order of the evaluation of the functional integral: first integrate out fluctuations, then expand in the sources. The integration over $B_{nn'}$ leads to a fluctuation factor $\tilde{F}$ similar to F_0 above, only that ϵ_{n_1} and ϵ_{n_2} are exchanged and coupled to the respective sources $\kappa_{1,2}$:

$$\tilde{F} = F_0\,\frac{\epsilon_{n_1}-\epsilon_{n_2}}{\epsilon^{\kappa}_{n_2}-\epsilon^{\kappa}_{n_1}}\prod_{n\neq n_1}\frac{\epsilon_n-\epsilon_{n_2}}{\epsilon_n-\epsilon^{\kappa}_{n_1}}\prod_{n'\neq n_2}\frac{\epsilon_{n_1}-\epsilon_{n'}}{\epsilon^{\kappa}_{n_2}-\epsilon_{n'}}\,. \qquad (3)$$

We next add to the products the "missing" factors $n = n_{1,2}$ and use that the now unconstrained products over n, n', as well as the factor F_0, give unity (causality). This leads to the result

$$\tilde{F} = \frac{\epsilon_{n_1}-\epsilon_{n_2}}{\epsilon^{\kappa}_{n_2}-\epsilon^{\kappa}_{n_1}}\,\frac{\epsilon_{n_1}-\epsilon^{\kappa}_{n_1}}{\epsilon_{n_1}-\epsilon_{n_2}}\,\frac{\epsilon^{\kappa}_{n_2}-\epsilon_{n_2}}{\epsilon_{n_1}-\epsilon_{n_2}} = \frac{\kappa_1\kappa_2}{\omega^{+2}}\,, \qquad (4)$$

where the last equality is valid to leading non-vanishing order in the sources, κ_1, κ_2. Notice the proportionality of $\tilde{F}$ to $\kappa_1\kappa_2$. This implies (a) that the sources in the action $S[\tilde{\Lambda}]$ may be set to zero (we are differentiating the functional at $\kappa_l = 0$) and (b) that *only* the non–standard saddle point $\tilde{\Lambda}$ contributes to the correlation function. Indeed, for any other non-standard saddle point one or several signatures corresponding to energy arguments $\epsilon_n \neq \epsilon_{n_1,n_2}$ are changed. These energies are not coupled to sources. Repeating the steps outlined above one finds that the B-integral around these saddle points gives zero (i.e. the $\kappa \to 0$ limit of the factor $\tilde{F}$ above). The same argument also shows that the non-standard saddle points do not contribute to $Z[0]$. Differentiating Eq. (4) w.r.t. $\kappa_{1,2}$ and adding the contribution of the standard saddle point, we obtain the well known result

$$R(\omega) = -\frac{1}{2}\,\mathrm{Re}\,\frac{1-\exp(2is)}{s^2} = -\left(\frac{\sin s}{s}\right)^2 \qquad (5)$$

for the two point correlation function of the zero mode theory. We finally mention that the inclusion of spatially fluctuating modes into the formalism (a) does not change the saddle point structure and (b), after Gaussian integration, leads to a renormalization $\tilde{F} \to \mathcal{D}(\omega)\tilde{F}$, where

$$\mathcal{D}(\omega) \equiv \prod_{\mathbf{q}\neq 0}\frac{(D\mathbf{q}^2)^2}{(D\mathbf{q}^2)^2+\omega^2} \qquad (6)$$

and $\mathbf{q}$ are the quantized momenta associated to fluctuations in a finite size system. Combining this with the contribution of Λ_0, we reproduce the familiar result [4,8,5] for the level statistics of the unitary disordered electron gas. Since this result was obtained in a saddle point approximation, its validity is restricted to energies $\omega \gg \Delta$. Indeed, in the opposite limit the fluctuation modes B become too light to be treated in the Gaussian approximation. The fact that the zero mode result Eq. (5) is actually exact for any ω is a "coincidence" (see however Ref. [10]), specific to the unitary ensemble.

Part II: The analysis above crucially relied on the causality *postulate*: summations over continuous functions of ϵ produce zero. As mentioned above, this feature could not be made explicit because the validity of the action (1) is limited to low energies. In this second part of the Letter we are going to construct an energetically enlarged formulation whereby large and small ϵ are treated on the same footing. This will make the causality property manifest. At the same time we will see why the pragmatic scheme employed above is sufficient for the calculation of low energy observables.

In order to not unnecessarily complicate the discussion we will formulate this part of the analysis for an N-dimensional random matrix theory (RMT) Hamiltonian $H = \{H_{\mu\nu}\}$ defined through the Gaussian correlation law $\langle H_{\mu\nu}\rangle = 0$ and $\langle H_{\mu\nu}H_{\nu'\mu'}\rangle = N^{-1}\delta_{\mu\mu'}\delta_{\nu\nu'}$. The advantage gained is that H has neatly defined universal large energy properties, i.e. that we will not need to consider the non-universal UV asymptotics of the free–electron Hamiltonian underlying Eq. (1). Later on we will argue that, as far as the present discussion is concerned, the specific modeling of the Hamiltonian is of no relevance. Further, to deal with a manifestly UV-regularized model, we compactify our energy variables. This can be done by discretizing the temporal Keldysh contours $\mathcal{C}^l$ to a lattice of small spacing δ_t. As a result, the energy variables $\epsilon_n \in \{\delta_\epsilon, 2\delta_\epsilon, \ldots, K\delta_\epsilon\}$, where $K \equiv 2\pi/(\delta_\epsilon\delta_t) \gg 1$.

The effective Keldysh action for the RMT model can be obtained by a straightforward adaptation of previous derivations of the σ–model for RMT Hamiltonians [12] to the specifics of the Keldysh σ-model. As an intermediate result one obtains the partition function $Z[\hat{\kappa}] = \int \mathcal{D}Q\exp\{iS[Q,\hat{\kappa}]\}$ where the action $iS[Q,\hat{\kappa}] = -N\left(\frac{1}{2}\,\mathrm{tr}\,(Q^2) - \mathrm{tr}\ln\left[\hat{z}+Q\right]\right)$ with $\hat{z} = \sigma_3\delta_t^{-1}(e^{-i\sigma_3\delta_t\hat{\epsilon}}-1)$ (σ_3 acts in the Keldysh 2×2 space). At this stage $Q = \{Q^{l,l'}_{\epsilon,\epsilon'}\}$ is a $2K$-dimensional matrix that has been introduced to decouple the H-averaged action [12] (no constraint $Q^2 = \mathbb{1}$ as yet). The unusual phase–type appearance of the energy argument is due to the time discretization. However, in the limit $\epsilon\delta_t \ll 1$, $\hat{z} \to -i\hat{\epsilon}$ and we retrieve the standard form of the RMT σ-model action [12]. We next subject the action to a saddle point analysis (stabilized by the parameter $N \gg 1$). Variation of the action w.r.t. Q yields the quadratic equation $Q = (\hat{z}+Q)^{-1}$. The 2^{2K}–fold degenerate energy diagonal set of solutions, $\Lambda^{(l)}(\epsilon_n) \equiv \Lambda^{l,l}_{\epsilon_n,\epsilon_n}$, is given by

$$\Lambda^{(l)}(\epsilon_n) = -\frac{1}{2}\Big[z_l(\epsilon_n) \pm i(-z_l^2(\epsilon_n)-4)^{1/2}\Big]. \qquad (7)$$

This expression determines the entire spectral structure of the model. First, notice that for energies $\epsilon_n \ll 1 \ll \delta_t^{-1}$, $\Lambda^{(l)}(\epsilon_n) = \pm 1 + \mathcal{O}(\epsilon_n)$. This means that the solutions $\Lambda^{(l)}(\epsilon_n)$ represent an UV extension of the saddle points Λ discussed in part I. We next ask whether the two sign alternatives in Eq. (7) are equivalent or whether

the model has a preferred choice. Indeed, the latter is the case: for energies $\epsilon_n \gg 1$ greatly in excess of the width of the spectrum, $\Lambda^{(l)}(\epsilon_n)$ must approach zero – the free Gaussian saddle point of the non–disordered model. This condition determines $\Lambda_0(\epsilon_n) = \frac{1}{2}[-\hat{z}(\epsilon_n) + \sigma_3(-\hat{z}^2(\epsilon_n) - 4)^{1/2}]$ as the canonical solution. For low energies $\Lambda_0(\epsilon_n)$ reduces to the saddle point Λ_0 discussed above. This saddle point has the important property, $\sum_n [\Lambda_0(\epsilon_n)]^k = 0$, k a positive integer. The outline of the proof is as follows: (due to the presence of a finite imaginary increment) the summation over ϵ_n is equivalent to an integration of the variable $w = \exp\{-i\delta_t \epsilon_n\}$ over the complex unit circle. It is straightforward to verify that for $|w| \geq 1$, $\Lambda_0(w)$ is analytic (the branch cut singularity of the square root lies *inside* the unit circle). Further, for $|w| \gg 1$, the integrand decays as $w^{-(k+1)}$. From Cauchy's theorem we conclude that the summation gives zero $\Box$. Summarizing, we have found a complex saddle point structure which extends the low energy saddle points $\Lambda = \pm 1$ discussed in part I into the UV regime. The complex structure of the theory entails the existence of a 'natural' saddle point Λ_0. Existence and behavior of Λ_0 vitally depend on the large energy, $\epsilon \gg 1$, asymptotics of the theory.

To more explicitly establish contact with the low energy regime discussed in part I, we next introduce fluctuations around the saddle points (7). Defining $Q = \Lambda + P$, where P is some Hermitian matrix [13], and expanding the action $S[\Lambda + P]$ to second order in P we obtain $iS^{(2)}_\Lambda[P] = -\frac{N}{2}\,\mathrm{tr}\,(P^2 + P\Lambda P\Lambda)$. This is the UV-extension of the low energy action discussed in part I. Indeed, substituting the small ϵ asymptotics of Λ and using that for the RMT model, $\Delta = \pi/N$ [12], we find that for energies $\epsilon \ll 1$, $S^{(2)}_\Lambda$ reduces to the action, Eq. (2).

One can now step by step repeat the analysis that led to the correlation function of part I. The only difference is that instead of energy denominators $\sim (\epsilon^+_n - \epsilon^-_{n'})$ constructions like $1 + \Lambda^{(1)}_0(\epsilon_n)\Lambda^{(2)}_0(\epsilon_{n'})$ appear. Due to the compact phase-type appearance of the energy arguments in $\Lambda_0(\epsilon_n)$ all energy summations converge. Further, the properties of Λ_0 discussed above imply the vanishing of expressions like $\sum_n f(\Lambda_0(\epsilon_n))$ where f may be any analytic function. This implements the causality principle. In parentheses we note that the detailed execution of this program yields a unit normalization of the partition function, without *any* undetermined prefactors.

The discussion above provokes the obvious question whether the conclusions drawn from the high energy asymptotics of the action are specific to the random matrix model? We believe that the answer is negative. Recapitulating the sequence of arguments, it is evident that everything hinges on the absence of singularities outside the complex unit circle defined through the compactified energy argument. This in turn is a guaranteed feature as long as the single particle retarded SCBA Green function of the model system has a well defined pole structure below the real axis. In practice, for condensed matter systems with non-universal high energy behavior, it may be difficult to find closed solutions of the mean field equations that manifestly display this feature. We believe, however, that this is a practical rather than a principle difficulty. Summarizing, the main goal of part II was to show that the causality feature underlying this and previous analyses of the effective Keldysh action can be made an explicit ingredient of the model, on the expense of including large energy asympotics. In practical applications of the formalism, one will normally use causality feature pragmatically, as exemplified in part I. Yet it cannot be excluded that situations arise, where large scale spectral structures become essential.

To conclude, on the example of level statistics, we have demonstrated how non–perturbative quantum effects may be incorporated into the framework of the dynamic Keldysh σ–model. In many respects, the Keldish scheme appears to be simpler and more transparent than its relatives, replica and SUSY. We expect our methods to be useful in the analysis of interaction phenomena in disordered electronic systems.

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[9] Within non–interacting problems the Keldysh saddle point block $\Lambda^{12}(\epsilon)$ (the distribution function) can be chosen at will. Without lost of generality we set $\Lambda^{12}(\epsilon) = 0$.

[10] M. R. Zirnbauer, cond-mat/9903338.

[11] Notice that contractions between $B_{n_1 n}$ and $B^\dagger_{n n_1}$ do not contribute. These terms lead to denominators $\propto (\epsilon^+_{n_1} - \epsilon^-_n)^{-1}$ which vanish after summation over n (causality).

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[13] We here write $\Lambda + P$ (instead of the common rotation ansatz $T\Lambda T^{-1}$) because to correctly describe the large energy structure of the model, fluctuation modes changing the eigen*values* of the saddle points ('massive modes' in the terminology of the σ–model) must be included.